Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

TRANSFORMATION GROUPS.

By DR. J. M. PAGE, Cobham, Va.

We propose in the following article to give an entirely elementary introduction to the theory of Groups of Infinitesimal Transformations, as that theory has been developed in recent years by Prof. Sophus Lie. The theory to which we wish to introduce the reader is of great importance in Differential Equations (for the treatment of which this theory furnishes the only logical ground), and also in Modern Geometry, Modern Algebra, and parts of Dynamics.

Although the arrangement, and a few of the methods, of this article, are new, the matter is virtually all Lie's. It has been collected from various scattering articles in the " Mathematische Annalen," the " Göttingen Nachrichten," the " Archiv for Mathematik og Naturindenskale " of Christiana; and from the writer's notes of lectures delivered by Lie in the winter of 1887. Since this article was planned, Lie has published the last of a series of masterly works on his theory, and it is the chief object of this elementary introduction to arouse the reader's interest in Lie's most important discoveries.

In the following we shall limit ourselves to two variables, unless expressly stated otherwise. Most of the developments, however, can be immediately extended to n variables.

A transformation of the points of the plane is given by a system of equations of the form

$$x_1 = \varphi(x, y), \quad y_1 = \psi(x, y), \tag{1}$$

where φ and ψ are analytical functions of their arguments, in Weierstrass's sense of the term.

We suppose here that the coordinate axes remain unchanged; but every point of general position, (x, y), is carried over to a new position, of which the coordinates are $\varphi(x, y)$ and $\psi(x, y)$.

If we solve (1) in the form

$$x = \Phi(x_1, y_1), \quad y = \Psi(x_1, y_1), \tag{2}$$

we obtain a transformation which would evidently carry the point $\varphi(x, y)$, $\psi(x, y)$ back to its original position (x, y). The transformation (2) is thus said to be *inverse* to the transformation (1).

The successive performance of (2) and (1) will evidently give us a transformation of the form

$$x_1 = x, \quad y_1 = y,$$

which we call the *identical* transformation. This transformation, therefore, really leaves the position of the point (x, y) unchanged.

Suppose now that we have a family of ∞^1 transformations, given by the equations

$$x_1 = \varphi(x, y, a) , \quad y_1 = \psi(x, y, a) , \tag{3}$$

where a is a parameter, which can assume ∞^1 continuous values. In general, then, it will *not* be the case that the result of performing any two transformations of the family (3) successively on the points of the plane will be equivalent to the result of performing a third transformation of the family (3) on those points. For instance, the equations

$$x_1 = a - x , \quad y_1 = y$$

represent a family of transformations, which do not possess the above peculiarity. For, if

$$x_2 = a_1 - x_1 , \quad y_2 = y_1$$

be a second transformation of the family, we find, when we perform the two successively on the point (x, y), that this point assumes a position given by

$$x_2 = a_1 - a + x , \quad y_2 = y .$$

But the transformation given by the last equations does not belong to our original family

$$x_1 = \text{const.} - x , \quad y_1 = y .$$

If now

$$x_1 = \varphi(x, y, a) , \quad y_1 = \psi(x, y, a)$$

be any given transformation of the family (3), and if

$$x_2 = \varphi(x_1, y_1, a_1) , \quad y_2 = \psi(x_1, y_1, a_1)$$

be a second transformation of that family, then the transformation which results from performing these two successively, evidently has the form

$$x_2 = \varphi\{\varphi(x, y, a), \psi(x, y, a), a_1\} , \quad y_2 = \varphi\{\varphi(x, y, a), \psi(x, y, a), a_1\} .$$

If it happens, now, that the right hand members of these equations have the general forms

$$\varphi\{\varphi(x, y, a), \psi(x, y, a), a_1\} \equiv \varphi\{x, y, \lambda(a, a_1)\} ,$$

$$\psi\{\varphi(x, y, a), \psi(x, y, a), a_1\} \equiv \psi\{x, y, \lambda(a, a_1)\} ,$$

where λ is a constant depending on a and a_1 alone, then we say that *the family of ∞^1 transformations* (3) *form a finite continuous group.*

In words expressed, our condition is evidently that *the result of performing successively any two transformations of the family* (3) *on the points of the plane, must be equivalent to the result of performing a third transformation of that family on those points.*

We shall add to the above, in the groups which we shall consider, the further condition that to every transformation with the parameter a of the family (3), we must be able to assign a certain transformation of the family, with the parameter α, such that the latter transformation shall be the *inverse* of the former. Here α is a function of a alone. From this further condition also follows immediately that the family (3) must contain the *identical* transformation.

Since our family (3) contains *one* arbitrary parameter, we call it, under the above conditions, a *group of one member;* or, symbolically, a G_1.

As an example of a G_1, we may take the family of ∞^1 rotations of the plane around a fixed point, which we shall take as the origin of coordinates. The well-known equations to these rotations are

$$\left.\begin{aligned} x_1 &= x\cos a - y\sin a\,, \\ y_1 &= x\sin a + y\cos a\,, \end{aligned}\right\} \tag{4}$$

where a is the angle through which we turn the radii vectores of the points of the plane.

A second transformation of the family (4) is given by the equations

$$x_2 = x_1\cos a_1 - y_1\sin a_1\,,$$

$$y_2 = x_1\sin a_1 + y_1\cos a_1\,.$$

By eliminating x_1 and y_1, we find

$$\begin{aligned} x_2 &= (x\cos a - y\sin a)\cos a_1 - (x\sin a + y\cos a)\sin a_1 \\ &= x(\cos a\cos a_1 - \sin a\sin a_1) - y(\sin a\cos a_1 + \cos a\sin a_1)\,, \end{aligned}$$

or

$$x_2 = x\cos(a + a_1) - y\sin(a + a_1)\,.$$

Similarly,

$$y_2 = x\sin(a + a_1) + y\cos(a + a_1)\,.$$

But the last two equations evidently represent a transformation of our family (4). Hence the two transformations with parameters a and a_1, performed successively upon the points of the plane, produce the same result as the performance of a single transformation of the same family with the parameter $(a + a_1)$. Also the *inverse* of a transformation with the parameter a is evidently the transformation with the parameter $-a$. We further find the *iden-*

tical transformation in our family by putting $a = 0$. Hence, according to our definition, the family (4) forms a group of one member, or a G_1.

If we have a family of transformations in the plane, the equations to which, of the form (3), contain r independent continuous parameters; and if the conditions are fulfilled, that the successive performance of any two transformations of the family is equivalent to the performance of a third transformation of the family, and that to every transformation of the family there is an *inverse* transformation in the family.—we then say that *the family of* ∞^r *transformations form a finite continuous group of r members, or a* G_r.

It is evident, for example, that the family of ∞^2 translations in the plane, of the form

$$x_1 = x + a\,, \quad y_1 = y + b\,, \tag{5}$$

form a G_2. For if,

$$x_2 = x_1 + a_1\,, \quad y_2 = y_1 + b_1$$

be a second translation of the family (5), then the successive performance of these two translations is equivalent to the performance of

$$x_2 = x + a + a_1\,, \quad y_2 = y + b + b_1\,.$$

Also, the translation with the parameter $-a$, $-b$ is always the *inverse* of the translation with the parameter a, b. Hence the family of ∞^2 translations (5) form a G_2.

Suppose, now, that we have a G_1 of the form

$$x_1 = \varphi(x, y, a)\,, \quad y_1 = \psi(x, y, a)\,. \tag{6}$$

Let us give the parameter a a fixed value e, and suppose that the corresponding transformation, (which we shall designate as (e), for brevity), carries the point of general position p over to the new position p_1. Then, by hypothesis, the inverse transformation in our G_1 (6), to (e), will carry the point p_1 back to p. Let us suppose that the parameter of the latter transformation is $\bar{e}$. A transformation, now, of which the parameter is of the form $\bar{e} + \delta e$, where δe is an infinitesimal quantity, will carry the point p_1 not exactly back to p, but to a position p', which is at an infinitesimal distance from p. If we perform (e) and $(\bar{e} + \delta e)$ successively, the result will be equivalent to the performance of a third transformation of our G_1: one that will take the point p directly to the position p'. But this must evidently be an *infinitesimal* transformation.

Let us carry this out analytically. The first transformation is represented by

$$x_1 = \varphi(x, y, e)\,, \quad y_1 = \psi(x, y, e)\,;$$

and the second by

$$x' = \varphi(x_1, y_1, \bar{e} + \delta e)\,, \quad y' = \psi(x_1, y_1, \bar{e} + \delta e)\,.$$

By eliminating x_1, y_1, we find the transformation which carries p directly to p'. It has the form

$$x' = \varphi\,\{\varphi(x, y, e), \psi(x, y, e), \bar{e} + \delta e\}, \quad y' = \psi\,\{\varphi(x, y, e), \psi(x, y, e), e + \delta e\}.$$

Develop in powers of δe, and we find

$$x' = \varphi\,\{\varphi(x, y, e), \psi(x, y, e), \bar{e}\} + \frac{\partial\varphi\{\varphi(x, y, e), \psi(x, y, e), \bar{e}\}}{\partial\bar{e}}\,\delta e + \ldots,$$

$$y' = \psi\{\varphi(x, y, e), \psi(x, y, e), \bar{e}\} + \frac{\partial\psi\{\varphi(x, y, e), \psi(x, y, e), \bar{e}\}}{\partial\bar{e}}\,\delta e + \ldots.$$

But, since the transformations (e) and $(\bar{e})$ are *inverse*, we have the identities

$$x \equiv \varphi\,\{\varphi(x, y, e), \psi(x, y, e), \bar{e}\},$$

$$y \equiv \psi\,\{\varphi(x, y, e), \psi(x, y, e), \bar{e}\};$$

and our above equations become

$$x' = x + \frac{\partial\varphi\,\{\varphi(x, y, e), \psi(x, y, e), \bar{e}\}}{\partial\bar{e}}\,\delta e + \ldots,$$

$$y' = y + \frac{\partial\psi\,\{\varphi(x, y, e), \psi(x, y, e), \bar{e}\}}{\partial\bar{e}}\,\delta e + \ldots;$$

and these equations evidently represent an *infinitesimal transformation.*

It is easy to see that the coefficients of δe, above, cannot vanish identically; for they may be written

$$\frac{\partial\varphi(x_1, y_1, \bar{e})}{\partial\bar{e}}, \quad \text{and} \quad \frac{\partial\psi(x_1, y_1, \bar{e})}{\partial\bar{e}};$$

and if these quantities were identically zero, that would necessitate the transformation (6) being free of any parameter, which is contrary to hypothesis.

Since $\bar{e}$ depends upon e alone, the equations to the infinitesimal transformation may evidently be written

$$x' = x + \xi(x, y, e)\,\delta e + \ldots,$$

$$y' = y + \eta(x, y, e)\,\delta e + \ldots;$$

and we see that every G_1 in the plane contains at least *one* infinitesimal transformation. In these equations we suppose e to be a mere member; but δe is an infinitesimal variable.

If, now, t be a parameter, the general form of an infinitesimal transformation in the plane will be

$$x_1 = x + \xi(x, y)\,\delta t + \ldots,$$

$$y_1 = y + \eta(x, y)\,\delta t + \ldots.$$

If we neglect, as usual, higher powers than the first of the infinitesimal δt, we may write the increments of x and y, under this infinitesimal transformation, in the form

$$\delta x = \xi(x, y)\,\delta t\,, \quad \delta y = \eta(x, y)\,\delta t\,.$$

It is evident that this transformation assigns to every point (x, y) a direction through which it is to be moved, given by

$$\frac{\delta y}{\delta x} = \frac{\eta(x, y)}{\xi(x, y)}\,;$$

and also it assigns to (x, y) an infinitesimal distance

$$\sqrt{\delta x^2 + \delta y^2} = \sqrt{\xi^2 + \eta^2}\,.\,\delta t\,,$$

through which it is to be moved.

We can get a clear and fruitful idea of an infinitesimal transformation, if we neglect terms of the second and higher powers of δt, and suppose that all the points of the plane are put into motion, simultaneously, by performing on them the infinitesimal transformation an infinite number of times. In this manner, a point (x, y) will assume a simply infinite number of continuous positions, which form a curve. The whole change of positions of the points of the plane, since it is repeated from moment to moment, may be called a *stationary motion;* and it may be compared with the *flow* of the molecules of a compressible fluid.

Let t represent the time, and let us measure it from a fixed point, say $t = 0$; and suppose that at that moment a certain molecule is at the position (x, y). Then, by means of the stationary motion, in the time t the molecule will be at a new position (x_1, y_1). These new coordinates x, y are functions of x, y, and t. If t increases by δt, then x_1 and y_1 receive the increments

$$\delta x_1 = \xi(x_1, y_1)\,\delta t\,, \quad \delta y_1 = \eta(x_1, y_1)\,\delta t\,;$$

and we may find x_1, y_1 as functions of t by integrating

$$\frac{\delta x_1}{\xi(x_1, y_1)} = \frac{\delta y_1}{\eta(x_1, y_1)} = \delta t\,, \tag{7}$$

with the condition that for $t = 0$, we have also $x_1 = x$ and $y_1 = y$

The integration will give us equations of the form

$$x_1 = \Phi(x, y, t)\,, \quad y_1 = \Psi(x, y, t)\,; \tag{8}$$

and our kinematic illustration easily shows that these equations represent a G_1. For, if our stationary motion, in the time t, carries the point (or molecule) (x, y) to the position (x_1, y_1); and in the time t, carries (x_1, y_1) to (x_2, y_2),—it is evident that in the time $t + t_1$ the point (x, y) will be carried to the position (x_2, y_2). Thus, our transformations (8) show the peculiarity of a G_1.

Lie has proved, by rigid analysis, that every infinitesimal transformation of the plane belongs to one and only one G_1, the finite transformations of which can be obtained by integrating equations of the form (7). These finite transformations can always be arranged into pairs of "inverses." Furthermore he has proved that every G_1 in the plane contains one and, virtually, only one infinitesimal transformation, where we consider two infinitesimal transformations which differ only by a numerical factor as virtually the same.

Similarly a G_r may be shown to contain r independent infinitesimal transformations. The finite transformations of a G_r (or the G_r itself) is said to be *generated* by performing each of the r independent infinitesimal transformations an infinite number of times.

It is easy to see that we may practically get the infinitesimal transformations of a G_r by assigning to the parameters values differing only by infinitesimal quantities from the values which give the identical transformation. As an example, we may put $a = \delta t$ in the G_1 of rotations, since $a = 0$ gives the identical transformation; and we see that the equations (4) become

$$x_1 = x - y\delta t\,, \quad y_1 = y + x\delta t\,,$$

which is an infinitesimal rotation.

Since our transformations are certain operations, which are independent of the coordinate axes, it is evident that if we have a set of equations which represent a G_1, they will still represent a G_1 if we introduce new variables.

A little consideration will show that the integration of the system (7) must give results of the form

$$\left.\begin{aligned} \Omega(x_1, y_1) &= \Omega(x, y)\,, \\ W(x_1, y_1) - t &= W(x, y)\,; \end{aligned}\right\} \qquad (9)$$

since, for $t = 0$, we must have $x_1 = x$ and $y_1 = y$. The equations (9) represent our G_1; and if we introduce new variables by means of the substitution

$$x_1' = \Omega(x_1, y_1), \quad x' = \Omega(x, y)\,,$$
$$y_1' = W(x_1, y_1), \quad y' = W(x, y)\,,$$

the equations to our G_1 take the form

$$x_1' = x'\,,$$
$$y_1' = y' + t\,.$$

But the last equations evidently represent a group of translations. Hence, *any G_1 in the plane, can, by a proper choice of variables, be brought to the form of a translation.*

We shall now derive a very useful symbol to represent an infinitesimal transformation.

If
$$x_1 = \varphi(x, y, t)\,, \quad y_1 = \psi(x, y, t)\,,$$
be the finite equations to a G_1, we can evidently consider any function $f(x_1, y_1)$ as a function of x, y, and t; and for the value of t which gives the identical transformation, say for the value $t = 0$, we must have $f(x_1, y_1) = f(x, y)$; and when t varies, $f(x_1, y_1)$ also varies. Thus we are led to ask for the increment δf which this function receives, when x_1 and y_1 receive their respective increments
$$\delta x_1 = \xi(x_1, y_1)\,\delta t\,, \quad \delta y_1 = \eta(x_1, y_1)\,\delta t\,.$$
We have
$$\delta f(x_1, y_1) \equiv \frac{\partial f(x_1, y_1)}{\partial x_1}\,.\,\delta x_1 + \frac{\partial f(x_1, y_1)}{\partial y_1}\,.\,\delta y_1$$
$$\equiv \left\{ \xi(x_1, y_1)\,\frac{\partial f(x_1, y_1)}{\partial x_1} + \eta(x_1, y_1)\,\frac{\partial f(x_1, y_1)}{\partial y_1} \right\} \delta t\,.$$
This is the increment of $f(x_1, y_1)$ under the infinitesimal transformation of our G_1. If, now, $t = 0$,
$$\delta f(x, y) \equiv \left\{ \xi(x, y)\,.\,\frac{\partial f(x, y)}{\partial x} + \eta(x, y)\,.\,\frac{\partial f(x, y)}{\partial y} \right\} \delta t\,.$$
It is obvious that if we know what increment a function $f(x, y)$ receives under the infinitesimal transformation of a G_1, we may consider the infinitesimal transformation, or the G_1, *known*. Hence we are led, quite naturally, to introduce the symbol
$$\xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y}\,,$$
to represent the infinitesimal transformation of a G_1. Thus, when we speak of the infinitesimal transformation,
$$-y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}\,,$$
we mean the transformation which assigns to x and y, respectively, the increments
$$\delta x = -y\delta t\,, \quad dy = x\delta t\,.$$
We shall represent our above symbol in a still shorter form by writing
$$X(f) \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y}\,.$$

We see, therefore, that $X(x)$ means, *put x in place of f in the above identity;* and similarly for $X(y)$. We find immediately

$$X(x) \equiv \xi\,, \quad X(y) \equiv \eta\,;$$

so that

$$X(f) \equiv X(x)\frac{\partial f}{\partial x} + X(y)\frac{\partial f}{\partial y}\,.$$

It is easy to see that if we introduce new variables x', y' into this symbol, we find

$$X(f) \equiv X(x')\frac{\partial f}{\partial x'} + X(y')\frac{\partial f}{\partial y'}\,.$$

Let us show, at once, the convenience of our new symbol, which we may write Xf. If we develop $f(x_1, y_1)$ by Maclaurin's formula, we find

$$f(x_1, y_1) \equiv f_1 \equiv \left\{ f(x_1, y_1) \right\}_{t=0} + \left\{ \frac{df_1}{dt} \right\}_{t=0} \cdot \frac{t}{1} + \left\{ \frac{d^2 f_1}{dt^2} \right\}_{t=0} \frac{t^2}{1\,.\,2} + \ldots .$$

Now

$$\frac{df_1}{dt} \equiv \frac{\partial f_1}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial f_1}{\partial y_1} \cdot \frac{dy_1}{dt}\,.$$

But

$$\frac{dx_1}{dt} = \xi(x_1, y_1)\,, \quad \frac{dy_1}{dt} = \eta(x_1, y_1)\,,$$

as usual; and, hence,

$$\frac{df_1}{dt} \equiv \xi(x_1, y_1)\frac{\partial f}{\partial x_1} + \eta(x_1, y_1)\frac{\partial f}{\partial y_1}\,.$$

But the last expression is nothing else than our symbol Xf, written in the variables x_1, y_1. Let us call this $X_1 f$. Hence

$$\frac{df_1}{dt} \equiv X_1 f.$$

Thus, any function f_1 of x_1 and y_1 gives, when differentiated with respect to t, $X_1 f$. But $X_1 f$ is itself such a function. Hence

$$\frac{d^2 f_1}{dt^2} \equiv \frac{d(X_1 f)}{dt} \equiv X_1(X_1 f)\,.$$

Similarly,

$$\frac{d^3 f_1}{dt^3} \equiv X_1(X_1(X_1 f))\,,$$

and the law of formation of our coefficients is now obvious.

If we put $t = 0$, then x_1 and y_1 are changed into x and y; $X_1 f$ becomes Xf; $X_1(X_1 f)$ becomes $X(Xf)$; etc. Thus we arrive at the important expansion

$$f(x_1, y_1) = f(x, y) + \frac{t}{1} Xf + \frac{t^2}{1 \cdot 2} X(Xf) + \dots \tag{10}$$

This holds, of course, in particular, for $f = x$, and $f = y$. Thus

$$\left.\begin{aligned} x_1 &= x + \frac{t}{1} \cdot X(x) + \frac{t^2}{1 \cdot 2} X(X(x)) + \dots, \\ y_1 &= y + \frac{t}{1} \cdot X(y) + \frac{t^2}{1 \cdot 2} X(X(y)) + \dots; \end{aligned}\right\} \tag{11}$$

and these are the finite equations of the G_1 of which

$$Xf \equiv \xi(x, y) \frac{\partial f}{\partial x} + \eta(x, y) \frac{\partial f}{\partial y}$$

is the infinitesimal transformation.

We see that (11) is only a new form of (9). In fact, (11) is the result of integrating the system (7) in an infinite series, with the condition that for $t = 0$, also $x_1 = x$, $y_1 = y$. Since x_1, y_1 are analytical functions of x, y, and t, it is certain that the series (11) converge for values of t near to $t = 0$.

Suppose now we demand that the function $\Omega(x, y)$ shall be *invariant* under all transformations of a given G_1. That is, analytically,

$$\Omega(x_1, y_1) = \Omega(x, y),$$

where the finite equations of our G_1 have the form

$$x_1 = \varphi(x, y, t), \quad y_1 = \psi(x, y, t),$$

and

$$Xf \equiv \xi(x, y) \frac{\partial f}{\partial x} + \eta(x, y) \frac{\partial f}{\partial y}$$

is the infinitesimal transformation. Thus, by (10), we must have

$$\Omega(x, y) + \frac{t}{1} \cdot X(\Omega) + \frac{t^2}{1 \cdot 2} X(X(\Omega)) + \dots = \Omega(x, y),$$

or, as an obviously necessary and sufficient condition,

$$X(\Omega) \equiv 0.$$

If this condition is fulfilled, $\Omega(x, y)$ is said to be the *invariant* of the G_1.

Every point (x, y) in the plane describes, when we perform on it the infinitesimal transformation of the G_1 an infinite number of times, a continuous curve, which we will call the *locus of the point under the* G_1. The direction in which the point (x, y) is moved, is always given by

$$\frac{\delta y}{\delta x} = \frac{\eta(x, y)}{\xi(x, y)}.$$

If now $\Omega(x, y)$ is the *invariant* of the G_1, Ω must satisfy the partial differential equation

$$X(\Omega) \equiv \xi \frac{\partial \Omega}{\partial x} + \eta \frac{\partial \Omega}{\partial y} = 0.$$

Hence the integral curves,

$$\Omega(x, y) = \text{const.},$$

will have, in each point, the tangential direction

$$\frac{\delta y}{\delta x} = \frac{\eta(x, y)}{\xi(x, y)}.$$

That is to say, the invariant of a G_1 in the plane, being put equal to a constant, will represent the curves in the plane, which we call the *loci* of the points of the plane under the G_1, or simply the *loci* of the G_1. It is evident that there are ∞^1 of these *loci*, one through every point of general position in the plane.

We see that any point, or points, in the plane for which

$$\xi(x, y) = \eta(x, y) = 0$$

are absolutely invariant under the infinitesimal transformation

$$Xf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y}.$$

A *family* of curves in the plane is said to be invariant under a G_1, when, under the transformations of the G_1, *every curve of the family is transformed to a curve of the same family.* The *loci* of a G_1 are evidently an invariant family; and every curve of this family is separately invariant.

If

$$w(x, y) = \text{const.}$$

be any family of curves in the plane, which, as a family, are invariant under a G_1 in the plane; and if

$$w(x_1, y_1) = \text{const.}$$

represents, as it may, the family, after the finite transformations of the G_1 have been performed upon it; then, since the family is invariant, a condition of the form

$$w(x_1, y_1) = \Omega(w(x, y))$$

must hold, by means of the finite transformations of the G_1. Here Ω may be any function of w.

Developing the left hand of the above equation in the usual manner, we find as a necessary and sufficient condition that the family of curves

$$w(x, y) = \text{const.}$$

shall be invariant under the G_1, Xf, is

$$X(w(x, y)) = \Omega(w(x, y)).$$

If a relation of this form holds, we say that the family of curves *admits of* the transformations of the G_1, Xf.

As an example, we see that if

$$Xf = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$$

(the G_1 of so-called *similitudinous* transformations), then the family of concentric circles

$$w \equiv x^2 + y^2 = \text{const.}$$

is invariant. For we find

$$X(w(x, y)) \equiv 2(x^2 + y^2) = 2w(x, y).$$

Thus the criterium holds.

We may mention that one of the most important methods of classifying groups is into *primitive* and *imprimitive.* A group of transformations in the plane, which leaves a family of ∞^1 curves invariant, is said to be *imprimitive.* If no family of ∞^1 curves is invariant, the group is *primitive.*

It is evident that every G_1 in the plane is *imprimitive,* since it always has a family of ∞^1 invariant *loci.*

For other methods of classification, we have to refer to Lie's works.

After discussing thus quite extensively the properties of the G_1 in the plane, we will say a few words about Lie's fundamental theorems for groups of more than one parameter, or member.

If X_1f and X_2f are two transformations of a G_r, we know that, by definition, the transformation which is equivalent to the successive performance of X_1f and X_2f also belongs to the G_r.

Let us represent by X_1f the infinitesimal transformation

$$x_1 = x + \xi(x, y)\,\delta t,$$

$$y_1 = y + \eta(x, y)\,\delta t;$$

and by X_2f

$$x_2 = x_1 + \xi_1(x_1, y_1)\,\delta\tau,$$

$$y_2 = y_1 + \eta_1(x_1, y_1)\,\delta\tau.$$

Here we know that δt and $\delta\tau$ are infinitesimals, of which the ratio $\delta t : \delta\tau$ is arbitrary.

Dropping infinitesimals of an order higher than the first, we find

$$x_2 = x + \xi(x, y)\,\delta t + \xi_1(x, y)\,\delta\tau\,,$$

$$y_2 = y + \eta(x, y)\,\delta t + \eta_1(x, y)\,\delta\tau\,;$$

and this new transformation, which we shall call X_3f, must belong to our group.

It is clear that, where λ and μ are certain finite quantities, and $\delta\varepsilon$ an infinitesimal, we may write

$$\delta t = \lambda\,.\,\delta\varepsilon\,, \quad \delta\tau = \mu\,.\,\delta\varepsilon\,,$$

and hence

$$x_2 = x + \{\lambda\,.\,\xi(x, y) + \mu\,.\,\xi_1(x, y)\}\,\delta\varepsilon\,,$$

$$y_2 = y + \{\lambda\,.\,\eta(x, y) + \mu\,.\,\eta_1(x, y)\}\,\delta\varepsilon\,.$$

From the last equations, we see that we may write

$$X_3f = \lambda\,.\,X_1f + \mu\,.\,X_2f\,.$$

Thus, we find that if a G_r contain X_1f and X_2f, it must also contain $\lambda\,.\,X_1f + \mu\,X_2f$, where the ratio $\lambda : \mu$ is arbitrary.

Similarly, we may show that, universally, if a G_r contain the r infinitesimal transformations $X_1f, X_2f, \ldots, X_rf$, it must contain every infinitesimal transformation of the general form,

$$\sum_1^r{}_i\,\lambda_i\,X_if\,. \qquad (\lambda_i \text{ constants})$$

It is easy to see the propriety of calling the r infinitesimal transformations *independent*, when no relation of the form

$$\sum_1^r{}_i\,\mu_iX_if = 0$$

can be established, where the μ_i are constants, not all of which are zero. In a G_r, there are always r *independent* infinitesimal transformations. In the finite equations to the G_r, as we saw, there are always r *independent* parameters.

We shall now develop another method for finding new transformations belonging to a group, of which we know certain transformations. Let S and T represent any two finite transformations of a G_r of the general forms

$$(S)\ x' = \varphi(x, y)\,, \qquad y' = \psi(x, y)\,;$$

$$(T)\ u' = \varphi_1(u, v)\,, \qquad v' = \psi_1(u, v)\,.$$

Suppose now that we perform the transformation T *upon* the transformation S. That is, we must introduce new variables into S in place of x', y' and x, y by means of T. This change will be accomplished by means of equations of the form

$$u = \varphi_1(x, y)\,, \qquad v = \psi_1(x, y)\,, \tag{12}$$

$$u' = \varphi_1(x', y')\,, \qquad v' = \psi_1(x', y')\,. \tag{13}$$

By solving (12) we find equations of the form

$$x = \Phi_1(u, v)\,, \quad y = \Psi_1(u, v)\,.$$

Hence, from the definition of S,

$$x' = \varphi\,\{\,\Phi_1(u, v)\,,\ \Psi_1(u, v)\}\,,$$

$$y' = \psi\,\{\,\Phi_1(u, v)\,,\ \Psi_1(u, v)\}\,.$$

Substituting now in (13), we get, finally,

$$\left.\begin{aligned} u' &= \varphi_1\,\{\varphi\,[\Phi_1(u, v),\ \Psi_1(u, v)]\,,\ \psi\,[\Phi_1(u, v)\,,\ \Psi_1(u, v)\,]\}\,, \\ v' &= \psi_1\,\{\varphi\,[\Phi_1(u, v),\ \Psi_1(u, v)]\,,\ \psi\,[\Phi_1(u, v)\,,\ \Psi_1(u, v)]\}\,. \end{aligned}\right\} \tag{14}$$

If, as is convenient, we indicate the successive performance of the two transformations S and T, by ST; and call the inverse transformation to T, T^{-1}; we easily see that the transformation (14) is made of the three, T^{-1}, S, and T, performed successively. That is, the transformation (14) may be written $T^{-1}S\,T$. Since S, T, and T^{-1} belong to our G_r, of course $T^{-1}S\,T$ is also a transformation of the G_r.

If, now, S and T are two finite transformations of a G_r, and Xf and Yf are the infinitesimal transformations of the G_r which *generate* respectively S and T, we wish to find the infinitesimal transformation, which generates the finite transformation $T^{-1}S\,T$.

Suppose that our transformations have the following general forms, respectively,

$$(S)\quad \begin{cases} x_1 = x + t\,.\,\xi(x, y) + \dfrac{t^2}{1\,.\,2}X(\xi) + \ldots\,, \\[2ex] y_1 = y + t\,.\,\eta(x, y) + \dfrac{t^2}{1\,.\,2}X(\eta) + \ldots\,; \end{cases}$$

$$Xf = \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y}\,;$$

$$(T)\quad\begin{cases} x' = x + \tau\,.\,\bar{\xi}(x, y) + \dfrac{\tau^2}{1\,.\,2}\,Y(\bar{\xi}) + \dots \\ \\ y' = y + \tau\,.\,\bar{\eta}(x, y) + \dfrac{\tau^2}{1\,.\,2}\,Y(\bar{\eta}) + \dots\,; \end{cases}$$

$$Yf \equiv \bar{\xi}(x, y)\frac{\partial f}{\partial x} + \bar{\eta}(x, y)\frac{\partial f}{\partial y}\,.$$

Now Xf is a transformation in the variables x, y. Suppose that, by means of T, we introduce into Xf the new variables x', y'; that is, we suppose, for the present, that T is a change of axes. Then we know that Xf assumes the form

$$X(x')\frac{\partial f}{\partial x} + X(y')\frac{\partial f}{\partial y}\,.$$

From the defining equations of the transformation T, we see

$$\left.\begin{aligned} X(x') &= X(x) + \tau\,X(\bar{\xi}) + \frac{\tau^2}{1\,.\,2}\,XY(\bar{\xi}) + \dots, \\ \\ X(y') &= X(y) + \tau\,X(\bar{\eta}) + \frac{\tau^2}{1\,.\,2}\,XY(\bar{\eta}) + \dots\,; \end{aligned}\right\} \qquad (15)$$

and we wish to express the right hand members of these equations in x' and y'.

Now $\xi(x, y)$ is, by means of T, a function of x', y'. Hence, by an obvious reversing of the general formula (10), we see that we may write

$$X(x) \equiv \xi(x, y) = \xi(x', y') - \tau\,Y(\xi(x', y')) + \frac{\tau^2}{1\,.\,2}\,YY(\xi') + \dots$$

$$X(y) \equiv \eta(x, y) = \eta(x', y') - \tau\,Y(\eta(x', y')) + \frac{\tau^2}{1\,.\,2}\,YY(\eta') \dots.$$

In the same way, we have

$$X(\bar{\xi}(x, y)) = X(\bar{\xi}(x', y')) - \tau\,YX(\bar{\xi}') + \dots,$$

$$X(\bar{\eta}(x, y)) = X(\bar{\eta}(x', y')) - \tau\,YX(\bar{\eta}') + \dots\,;$$

and

$$XY(\bar{\xi}(x, y) = XY(\bar{\xi}(x', y') - \dots$$

$$XY(\bar{\eta}(x, y) = XY(\bar{\eta}(x', y')) - \dots$$

We are now ready to find the right hand members of equations (15), in the new variables x', y'; and hence our new transformation

$$X(x')\frac{\partial f}{\partial x'} + X(y')\frac{\partial f}{\partial y'}\,.$$

By carrying out the substitutions, we find for our new transformation

$$\xi(x', y')\frac{\partial f}{\partial x'} + \eta(x', y')\frac{\partial f}{\partial y'} - \tau\Big[(X(\bar{\xi}') - Y(\xi'))\frac{\partial f}{\partial x'} + (X(\bar{\eta}') - Y(\eta'))\frac{\partial f}{\partial y'}\Big]$$

$$+ \frac{\tau^2}{1.2}\Big[(YY(\xi') - 2YX(\bar{\xi}') + XY(\xi'))\frac{\partial f}{\partial x'}$$

$$+ (YY(\eta') - 2YX(\bar{\eta}') + XY(\bar{\eta}'))\frac{\partial f}{\partial y'}\Big] + \dots .$$

Let us now, for brevity, indicate, with Poisson, by the symbol, (X, Y), the operation

$$X(Yf) - Y(Xf).$$

We then easily see that the coefficient of τ in the above transformation is exactly (X, Y), written in the variables x', y'. A little consideration will also show that, if the expression (X, Y) is itself combined with Yf, just as Xf and Yf are in the bracket expression (X, Y), the resulting expression, of the form $((X, Y), Y)$, will be the coefficient of $\frac{\tau^2}{1.2}$.

As nothing depends upon the accented variables, we may drop the accents, and say that the infinitesimal transformation which generates $T^{-1} S T$ has, as far as the first three terms of the series, the general form

$$Xf + \tau(X, Y) + \frac{\tau^2}{1.2}((X, Y), Y) + \dots .$$

But since Xf is a transformation of our G_r, we see by subtraction that

$$\tau(X, Y) + \frac{\tau^2}{1.2}((X, Y), Y) + \dots$$

must belong, also, to our G_r. Divide out by τ, and then put $\tau = 0$, and *we see that*

$$(X, Y)$$

must belong to our group, if Xf and Yf do.

Suppose now that

$$X_1 f, \; X_2 f, \; \dots, \; X_r f$$

are the r independent infinitesimal transformations of our G_r. Then every infinitesimal transformation of the G_r is included in the general form

$$\sum_1^r{}_i \lambda_i X_i f. \qquad (\lambda_i = \text{const.})$$

But if our G_r contain X_if and X_kf, we have just seen, above, that it must also contain (X_i, X_k). *Hence, for all the infinitesimal transformations of our* G_r, *relations must hold of the general form*

$$(X_if,\, X_kf) = \overset{r}{\underset{1}{\Sigma}}{}_i\, c_{iks}\, X_sf\,, \qquad (i, k = 1, 2, \ldots, r) \quad (16)$$

the c_{iks} *being constants.* The equations (16) express analytically the *fundamental theorem* of Lie's works.

It can be shown that, *if r independent infinitesimal transformations satisfy the relations* (16), *they will then generate* (*or form*) *a* G_r. This definition of a G_r can be shown to be identical with that given at the beginning of this article, where we made use of the *finite* equations of the G_r.

For any *three* equations of the form X_if, X_kf, and X_jf, the celebrated Jacobian identity must hold good. Thus, we must have

$$((X_i, X_k), X_j) + ((X_k, X_j), X_i) + ((X_j, X_i), X_k) \equiv 0\,. \qquad (17)$$

As Engel has remarked, we may verify this identity by merely performing the operations indicated. Thus

$$((X_i, X_k), X_j) \equiv X_i(X_k(X_j)) - X_k(X_i(X_j)) - X_j(X_i(X_k)) + X_j(X_k(X_i))\,.$$

If we write out $((X_k, X_j), X_i)$ and $((X_j, X_i), X_k)$ in full in a similar manner, we find that the terms of (17) will all cancel in pairs.

We see, thus, that the c_{iks} in (16) are not arbitrary, but must fulfill certain conditions which flow from Jacobi's identity. These constants are called the *setting* of the G_r, and one of the problems of Lie's theory is *to find all groups of a given setting.*

We may remark that Lie has found all groups in one and two variables; and also the most interesting ones in three variables. We hope to give some of the more interesting and elementary of these results in a subsequent paper.